

Dynamical degrees of freedom for higher genus Riemann surface in (2+1)-dimensional general relativity

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Abstract

A homogeneous two-dimensional metric including the degrees of freedom of Teichmüller deformation is developed. The Teichmüller deformation is incorporated by affine stretching of complex structure. According to Yamada's investigation by pinching parameter, concrete formulation for a higher genus Riemann surface can be realized. We will have a homogeneous standard metric including the dynamical degrees of freedom as Teichmüller deformation in a leading order of the pinching parameter, which would be treated as homogeneous anisotropic metric for a double torus universe, which satisfy momentum constraints.

1 Introduction

In spite of the absence of propagating gravitational wave, the gravitation in (2+1)-dimensional spacetime has been investigated from the various viewpoints[1, 2, 3, 4, 5], and its importance is still now surviving supported by the new concepts[6, 7, 8]. Probably, that will be recognized as the result of that there still remain global degrees of freedom of the spacetime in (2+1)-dimensional gravity.

As pointed out by Hosoya, Nakao and Moncrief[3], it seems clear that the two-dimensional spatial geometry is understood as a Riemann surface with complex structure in York's timeslicing[9]. In other words, in (2+1)-dimensional spacetime, a spacelike hypersurface is locally conformally flat, and then the 'geometry' is represented by the conformal class of itself. That will be corresponding to the complex structure, in the identification between the conformal transformation and bi-holomorphic transformation. To investigate such a complex structure on a Riemann surface one may conventionally prefer to a local complex coordinate (z, \bar{z}) , $z = x + iy$. The Riemann surface is a two-dimensional surface with complex structure, which is with a set of local complex coordinate, recognizing an equivalence relation by bi-holomorphic function. In a sense of Riemannian geometry, the complex structure is just an equivalence class of Riemannian two-geometry by a conformal mapping. We would realize the degrees of freedom of the complex structure by the Teichmüller deformations parametrized by parameters in Teichmüller space[11, 12].

In [3], it was shown that the transverse traceless part of extrinsic curvature \tilde{K}_{ab} is a holomorphic quadratic differential spanning a complex vector space of $(3g - 3)$ dimensions and one dimension exceptionally for a torus. Determining the holomorphic quadratic differential for a torus universe to solve the momentum constraint, the Hamiltonian constraint would be resolved for pure gravity on homogeneous spacelike hypersurface in York's gauge choice. Moreover, these aspects can be reformulated[4], considering a standard homogeneous metric containing the global degrees of freedom as Teichmüller parameters of torus in the context of affine stretching[12]. When we consider a homogeneous spatial hypersurface by the definition of time coordinate, the Einstein equations will become an ordinary differential equation about time of the expansion coefficients of the Teichmüller deformation by the holomorphic quadratic differential. For a higher genus $g > 1$ Riemann surface, however, similar analysis is not straightforward, and the holomorphic quadratic differential accounts for its difficulty. There, that is not a constant and also is not given in concrete expression, remarkably containing $4g - 4$ zeros. So, it seems difficult to get a homogeneous standard metric such that the Einstein equation is reduced to ordinary differential equations of time.

In the present study, we attempt to treat the complex structures by the expansions around the Riemann surface with highly pinching narrow bridges. Yamada[10] have established the description of such an expanded complex structure in full order of the pinching parameter ϵ . We will have concrete expression of holomorphic quadratic differential in the expansion. Especially in leading order of ϵ , we will

determine the homogeneous standard metric to study the pure gravitational dynamics of a double torus universe.

First of all we give a formal definition of the Teichmüller deformation by holomorphic quadratic differential in the context of the affine stretching in the second section. The leading order of holomorphic quadratic differential by the pinching parameter is developed for a double torus according to Yamada's work in the third section. The fourth section provides a homogeneous standard metric of it and the final section is devoted for summary and discussions.

2 Affine stretching

Our purpose of the present study is to formulate the geometrical degrees of freedom in (2+1)-dimensional gravity where spatial section of the spacetime realized as a higher genus Riemann surface. From the view point of metric tensor, the Riemann surface is the conformal class of the Riemannian geometry on two dimensional manifold which is a spacelike section Σ of the spacetime because complex local coordinate $\{z_\alpha\}$ clarify that conformal mapping is identified to bi-holomorphic function.

As in an appropriate gauge choice, the time covariance would be recognized as in the conformal degrees of freedom[4, 5] and then the geometrical degrees of freedom will be identified to those of the Riemann surface. Nevertheless, since two-dimensional Riemannian geometry admitting a locally conformally flat coordinate, the local degrees of freedom are absence. (That implies there is no gravitational wave.) We will account for the global degrees of freedom, which is regarded as Teichmüller deformation[12].

The Teichmüller deformation will be realized as a quasiconformal mapping of the complex structure $(R, \{z_\alpha\})$ of the Riemann surface R . Then the degrees of freedom is represented by the Beltrami coefficient $\mu = f_{,\bar{z}}/f_{,z}$ of the quasiconformal mapping $w = f(z, \bar{z})$ since the pull back of the local metric function $|dw|^2$ is given by $|f_{,z}|^2|dz + \mu d\bar{z}|^2$, and we are required to solve Beltrami equation. With vanishing μ , f is a bi-holomorphic function (\sim conformal mapping). One may call μ complex dilatation which represents ellipticity and rotation of the infinitesimal circular image.

Another way to realize the degrees of freedom for Teichmüller deformation is to consider affine transformation of the local complex coordinate[12]. In the following, we see a two dimensional metric including the degrees of freedom of the Teichmüller deformation is given as a result of affine stretching along the coordinate incorporated by the holomorphic quadratic differential. The affine transformation of the local coordinate $z \mapsto w(z, \bar{z}) = \alpha z + \beta \bar{z}$, ($\alpha, \beta \in \mathbb{C}$, $|\alpha| > |\beta|$) reduces to the

$$z \mapsto z + k\bar{z} \quad (1)$$

by the redefinition of the local coordinate z, w and assign the ellipticity by $0 < k = \beta/\alpha < 1$, which will be real by the rotation of the complex coordinates and fixed.

Of course, the local coordinate can be extended by analytic continuation of bi-holomorphic mapping. Then the pull back of the Euclidean metric $|dw|^2$ by the bi-holomorphic mapping $z = h(\zeta)$ is given by

$$|dw|^2 = |h'|^2|d\zeta + k \frac{\bar{h}'}{|h'|^2} d\bar{\zeta}|^2. \quad (2)$$

Here $\varphi \equiv (h')^2$ is regarded as a holomorphic quadratic differential since it is transformed by the change of local coordinate $(U_j, z_j) \mapsto (U_k, z_k)$ as $\varphi_k(z_k) = \varphi_j(z_j)(dz_j/dz_k)^2$. Therefore the quasiconformal mapping is determined by a holomorphic quadratic differential φ and k .

It is well known from the Riemann Roch Theorem[11], that the holomorphic quadratic differential spans a complex vector space $A_2(R)$ whose dimension is $3g - 3$. In the context of Riemannian geometry the space of complex structure is identified to conformal class of Riemannian geometry on the surface Σ , and $A_2(R)$ is homeomorphic to Teichmüller space $T(R)$ (the space of Teichmüller deformation or quasi conformal mapping)

$$T(R) = \frac{\mathcal{M}}{\{\text{Conf}(\Sigma)\}\{\text{Diffeo}(\Sigma)\}},$$

where \mathcal{M} is the space of Riemannian geometry on Σ to become quotient by the conformal equivalence and diffeomorphism¹. Then considering a quasiconformal mapping f which is affine transformation in a local coordinate z as (1), its Beltrami coefficient is represented by an element φ of $A_2(R)$ as

$$\mu_f = k \frac{\bar{\varphi}}{|\varphi|},$$

which is a quasiconformal mapping as Teichmüller deformation for $(k, \varphi)^2$.

These aspects roughly illustrated in Figure 1. Since $\varphi d\zeta^2$ is holomorphic quadratic differential, its root ($\sim h'd\zeta$) gives holomorphic one-form dz . Then the integration of it $z = \int h'd\zeta$ gives not a global coordinate but horizontal (or vertical) foliation [13] by analytic continuation, since such a local flat coordinate cannot be extended to global one beyond the zeros of φ as suggested by the uniformization theorem. In other words, the horizontal foliation has branches at the zeros as illustrated. Then the local Euclidean metric of local coordinate (z, \bar{z}) is affine stretched by (1).

The equivalence class of this affine stretched complex structure is given by the conformal class of the metric (2). Usually we will choose a conformally invariant one $\tilde{g}_{ab} := g_{ab}/\sqrt{g}$ as a representative of the conformal class of the affine stretching:

$$|\widetilde{dw}|^2 = |d\zeta + k \frac{\bar{\varphi}}{|\varphi|} d\bar{\zeta}|^2. \quad (3)$$

Consequently, the Teichmüller deformation generating the Teichmüller space $T(R)$. (For the case of torus, exceptionally the dimension of $A_2(R)$ is one and since the μ_f is essentially constant, the global affine transformation uniformly gives a metric function $|dw|^2$ written by the well known Teichmüller parameter τ for a torus[4].)

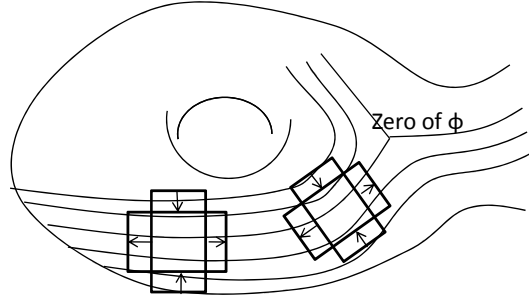


Figure 1: A horizontal foliation which is introduced by the root of a holomorphic quadratic differential φ is illustrated. That indicates the direction of the affine stretching by φ .

To have the holomorphic quadratic differential, we consider holomorphic one-form firstly. For a Riemann surface R of genus g whose canonical homology basis are given by $(A_1, \dots, A_g, B_1, \dots, B_g)$, there exists g holomorphic one-forms ν_i normalized as

$$\int_{A_i} \nu_j = \delta_{ij}. \quad (4)$$

These forms can be neatly encapsulated in a unique singular bilinear two-form[10]

$$\omega(u, v) = \left\{ \frac{1}{(u-v)^2} + \text{regular terms} \right\} du dv \quad (5)$$

with normalization

$$\int_{A_i} \omega(u, \cdot) = 0 \quad (6)$$

¹Here we do not consider modular transformation. On considering quantum theory, we should redefine the modular space as the modular class of the Teichmüller space[5].

² k is a fixed parameter, which will have any technical role in formulation of canonical variables in our forthcoming work[15]

and this is a differential of second kind, where u, v is complex local coordinates.

From the Riemann bi-linear relations, we see

$$\nu_i(u) = \frac{1}{2\pi i} \int_{B_i} \omega(u, \cdot). \quad (7)$$

Then the quadratic differential spans a complex vector space,

$$A_2(R) = \langle \varphi_{ij} \rangle_{\mathbb{C}} = \langle \nu_i \cdot \nu_j | i, j = 1, \dots, g \rangle_{\mathbb{C}}. \quad (8)$$

For a higher genus ($g > 1$) Riemann surface, this is $3g - 3$ dimensional from the Riemann Roch theorem. From the Teichmüller's theorem[12], the Teichmüller space $T(R)$ is homeomorphic to $A_2(R)^3$ and furthermore to \mathbf{R}^{6g-6} .

As an example, we will consider the case of a torus. So, different from the higher genus, there is only one independent holomorphic one-form $\nu = d\zeta$, which is constant. Then the quadratic form $(h')^2 d\zeta^2$ is given by two real constant parameters as $(\varphi_R + i\varphi_I)d\zeta^2$. In other words,

$$h'd\zeta = (\nu_R + i\nu_I)d\zeta$$

is complex and constant. Consequently the affine stretched metric (2) is given by

$$|dw|^2 = |(\nu_R + i\nu_I)d\zeta + (\nu_R - i\nu_I)kd\bar{\zeta}|^2 \quad (9)$$

$$= (d\zeta_x, d\zeta_y) \begin{pmatrix} (1+k)\nu_R^2 + (1-k)\nu_I^2 & -2k\nu_R\nu_I \\ -2k\nu_R\nu_I & (1+k)\nu_I^2 + (1-k)\nu_R^2 \end{pmatrix} \begin{pmatrix} d\zeta_x \\ d\zeta_y \end{pmatrix}. \quad (10)$$

Introducing a conventional Teichmüller parameters, instead of (ν_R, ν_I)

$$\xi = \frac{-2k\nu_R\nu_I}{(1-k)\nu_R^2 + (1+k)\nu_I^2}, \quad \eta = \frac{\sqrt{1-k^2}(\nu_R^2 + \nu_I^2)}{(1-k)\nu_R^2 + (1+k)\nu_I^2},$$

the conformally invariant representative $\tilde{g}_{ab} = g_{ab}/\sqrt{g}$ gives well known standard metric for torus[4, 12];

$$\frac{1}{\eta} \cdot \begin{pmatrix} \xi^2 + \eta^2 & \xi \\ \xi & 1 \end{pmatrix}. \quad (11)$$

For vanishing k , $|dw|^2$ reduces to the conformally Euclidean form $|\varphi||d\zeta|^2$ meaning equivalent complex structure to the original one.

3 pinching parameter and complex structure

It may be a consensus that it is strongly difficult to give a concrete expression of the differential forms $\omega(u, v), \nu_i, \varphi_{ij}$ for a general Riemann surface with higher genus $g > 1$.

On the other hand, for a genus g Riemann surface including a narrow bridge structure, one can approximately compose the differentials $\omega(u, v), \nu_i, \varphi_{ij}$ from the differentials on two Riemann surfaces with lower genus, from which the genus g Riemann surface is obtained by sewing. Therefore, we only consider the case where a Riemann surface can be thought of as one composed of two Riemann surfaces connected by a narrow bridge. A general method for calculating $\omega(u, v)$ for any two sewn Riemann surfaces has been given by Yamada [10].

In general, two compact Riemann surfaces S_1 and S_2 of genus g_1 and g_2 can be sewn together, giving a Riemann surface of genus $g_1 + g_2$. Choose complex local coordinates z_a on S_a ($a = 1, 2$), and excise the two disks $|z_a| < |\epsilon|$, where ϵ is a complex parameter satisfying $|\epsilon| < 1$. The centers of disks are taken at a point $z_a = 0$. The two surfaces are sewn together by identifying the annular regions $|\epsilon| \leq |z_a| \leq 1$ via the relation

$$z_1 z_2 = \epsilon.$$

³To be exact, $A_2(R)_1 = \{\varphi \in A_2(R) | \|\varphi\|_1 < 1\}$ ($\|\varphi\|_1 := 2 \int_R |\varphi(z)| dx dy$) is homeomorphic to $T(R)$.

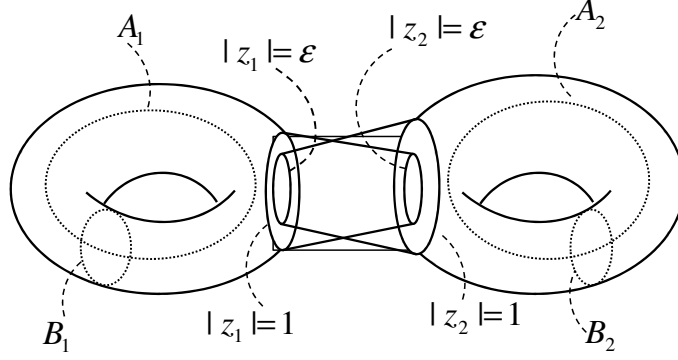


Figure 2: Two tori are sewn together by identifying the annular regions $|\epsilon| \leq |z_a| \leq 1$ via the relation $z_1 z_2 = \epsilon$.

By the way, for a pinching parameter ϵ , $\mathcal{A}_a = \{\epsilon \leq |z_a| \leq 1\} \subset S_a$ is a cylinder to be identified. Let $\mathcal{C}_a(z_a) \subset \mathcal{A}_a$ denote a simple closed, anti-clockwise oriented contour parametrized by z_a surrounding the puncture at $z_a = 0$, which is in relation of sewing $\mathcal{C}_1(z_1) \sim -\mathcal{C}_2(z_2)$ via $z_1 z_2 = \epsilon$. In the following, we refer and adopt following three results of [10], without proof.

Yamada[10] had started from the following lemma.

Lemma 1.

$$\omega(u, v) = \omega^{(a)}(u, v) \delta_{ab} + \frac{1}{2\pi i} \oint_{\mathcal{C}_a(z_a)} \left(\omega(v, z_a) \int_0^{z_a} \omega^{(a)}(u, \cdot) \right) \quad (12)$$

for $u \in S_a, v \in S_b (a, b = 1, 2)$.

$\omega^{(a)}$ is a unique bilinear two-form for a Riemann surface S_a .

Then it was demonstrated that the following expression expanded by the power of ϵ will be possible.

Lemma 2.

$$\omega(u, v) = \begin{cases} \omega^{(a)}(u, v) + a_a(u) X_{\bar{a}a} a_a^T(v) & u, v \in S_a \\ a_a(u) (-I + X_{\bar{a}a}) a_a^T(v) & u \in S_a, v \in S_{\bar{a}} \end{cases} \quad (13)$$

where $a_a(u)[k]$ is row vector and $X_{ab}[k, l]$ is matrix indexed by $k, l = 1, 2, \dots$, defined by

$$X_{ab}[k, l] = \frac{\epsilon^{(k+l)/2}}{\sqrt{k!l!}} \frac{1}{(2\pi i)^2} \oint_{\mathcal{C}_a(u)} \oint_{\mathcal{C}_b(v)} u^{-k} v^{-l} \omega(u, v). \quad (14)$$

$$a_a(u)[k] = \frac{\epsilon^{k/2}}{2\pi i \sqrt{k!}} \oint_{\mathcal{C}_a(z_a)} z_a^{-k} \omega^{(a)}(u, z_a), \quad (15)$$

where \bar{a} means the complement of a .

Besides, X_{ab} is given in terms of

$$A_a[k, l] = \frac{\epsilon^{k/2}}{2\pi i \sqrt{k!}} \oint_{\mathcal{C}_a(u)} u^{-k} a_a(u)[l]. \quad (16)$$

The consistency is guaranteed by this proposition.

Proposition 1.

$$X_{aa} = A_a (I - A_{\bar{a}} A_a)^{-1} \quad (17)$$

$$X_{a\bar{a}} = I - (I - A_a A_{\bar{a}})^{-1} \quad (18)$$

Here,

$$(I - A_a A_{\bar{a}})^{-1} = \sum_{n \geq 0} (A_a A_{\bar{a}})^n \quad (19)$$

and is convergent as a power series in ϵ for $|\epsilon| < 1$.

Though the full expression is complicated, the leading order of contribution of the pinching parameter ϵ can be simply determined. Now we calculate the leading contribution of ϵ from the lemma's and the proposition for a $g = 2$ Riemann surface (double torus) with the pinching parameter. S_1 and S_2 are tori and their Teichmüller parameters are τ_1 and τ_2 , respectively.

First of all, we remind the definitions of the Weierstrauss's zeta function $\zeta(z)$ and elliptic function $\mathcal{P}(z)$ [11], for a torus with a Teichmüller parameter τ ;

$$\zeta(z) = \frac{1}{z} + \sum_{m, n \in \mathbb{Z}, m^2 + n^2 \neq 0} \left[\frac{1}{z - m - n\tau} + \frac{1}{m + n\tau} + \frac{z}{(m + n\tau)^2} \right], \quad (20)$$

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{m, n \in \mathbb{Z}, m^2 + n^2 \neq 0} \left[\frac{1}{(z - m - n\tau)^2} - \frac{1}{(m + n\tau)^2} \right]. \quad (21)$$

In (21) the last term of summation $\sum (m + n\tau)^{-2} = E_2(\tau)$ is convergent and known as one of the Eisenstein series $E_{2k} = \sum \frac{1}{(m + n\tau)^{2k}}$, such that the zeta function will be expanded as $\zeta(z) = \frac{1}{z} - \sum_{k=1}^{\infty} E_{2k+2} z^{2k+1}$. From $\mathcal{P}(z) = -\frac{d}{dz}\zeta(z)$, also $\mathcal{P}(z)$ can be expanded as $\mathcal{P}(z) = \frac{1}{z^3} + \sum_{k=1}^{\infty} (2k+1)E_{2k+2} z^{2k}$.

By definition, it is obvious that $\mathcal{P}(z)$ is periodic function as

$$\mathcal{P}(z) = \mathcal{P}(z + m + n\tau), \quad m, n \in \mathbb{Z}. \quad (22)$$

Integrating it, one may realize that $\zeta(z)$ is with quasi-periodic properties

$$\zeta(z) = \zeta(z + m + n\tau) - (\eta_1 m + \eta_2 n) \quad m, n \in \mathbb{Z}. \quad (23)$$

With the aid of Legendre's equation $\eta_1 \tau - \eta_2 = 2\pi i$, we conclude that $\eta_1 = E_2$ and $\eta_2 = \tau E_2 - 2\pi i$. Defining $P_1(\tau, z) = \zeta(z) - z E_2(\tau)$, the periodicity is simply expressed as $P_1(\tau, z) = P_1(\tau, z + m + \tau n) + 2\pi i n$, $m, n \in \mathbb{Z}$.

From (5), $\mathcal{P}(z)$ will provide a unique bilinear two-form (which is differential of second kind) for a torus parametrized in the usual way but not normalized. We see

$$\omega^{(a)}(u, v) = (\mathcal{P}(\tau_a, u - v) + E_2(\tau_a)) du dv =: P_2(\tau_a, u - v) du dv \quad (a = 1, 2), \quad (24)$$

satisfies (6) by the quasi-periodicity of $P_1(z)$, (c.f. $P_2 = -\frac{d}{dz}P_1$), where τ_a is the Teichmüller parameter of each torus ($\sim S_a$).

Now we are back to genus 2 Riemann surface. From (15) and (16), one may have contributions of leading order of ϵ ,

$$a_a(u)[1] = \frac{\epsilon^{1/2}}{2\pi i} \oint_{\mathcal{C}_a(z_a)} z_a^{-1} \omega^{(a)}(u, z_a) = \epsilon^{1/2} P_2(\tau_a, u) du \quad (25)$$

$$A_a[1, 1] = \frac{\epsilon^{1/2}}{2\pi i} \oint_{\mathcal{C}_a(u)} u^{-1} a_a(u)[1] = \epsilon E_2(\tau). \quad (26)$$

Then a basis of our analysis is given;

$$\omega(u, v) = \begin{cases} P_2(\tau_a, u - v) du dv + \epsilon^2 E_2(\tau_a) P_2(\tau_a, u) P_2(\tau_a, v) du dv, & (u \in S_a, v \in S_a) \\ -\epsilon P_2(\tau_a, u) P_2(\tau_{\bar{a}}, v) du dv, & (u \in S_a, v \in S_{\bar{a}}) \end{cases}. \quad (27)$$

Deductively, we see the holomorphic one-form for the double torus with pinching parameter ϵ ,

$$\nu_1(u \in S_1) = \frac{1}{2\pi i} \int_{B_1(v)} [P_2(\tau_1, u-v) + \epsilon^2 E_2(\tau_1) P_2(\tau_1, u) P_2(\tau_1, v)] du dv \quad (28)$$

$$= -\frac{1}{2\pi i} [P_1(\tau_1, u-v+\tau_1) - P_1(\tau_1, u-v)] du \quad (29)$$

$$- \frac{1}{2\pi i} \epsilon^2 E_2(\tau_2) P_2(\tau_1, u) [P_1(\tau_1, v+\tau_1) - P_1(\tau_1, v)] du \quad (30)$$

$$= du + \epsilon^2 E_2(\tau_2) P_2(\tau_1, u) du \quad (31)$$

$$\nu_1(u \in S_2) = -\frac{1}{2\pi i} \int_{B_1(v)} \epsilon P_2(\tau_2, u) P_2(\tau_1, v) du dv \quad (32)$$

$$= -\epsilon P_2(\tau_2, u) du. \quad (33)$$

As expected, they coincides to that of torus with vanishing ϵ . Then we can identify its complex structure from them. For example[12], the holomorphic one-form determines a period matrix by

$$\int_{A_j} \nu_i = \delta_{ij} \quad (34)$$

$$\Omega_{ij} = \int_{B_j} \nu_i \quad (35)$$

$$= \left(\begin{array}{c} \int_{B_1} \nu_1 = \tau_1 + 2\pi i \epsilon^2 E_2(\tau_2) \\ \int_{B_1} \nu_2 = -2\pi i \epsilon \end{array} \middle| \begin{array}{c} \int_{B_2} \nu_1 = -2\pi i \epsilon \\ \int_{B_2} \nu_2 = \tau_2 + 2\pi i \epsilon^2 E_2(\tau_1) \end{array} \right). \quad (36)$$

The holomorphic quadratic differentials can be given by $\varphi_{ij} = \nu_i \nu_j$,

$$\varphi_{11} = \begin{cases} (1 + 2\epsilon^2 E_2(\tau_2) P_2(\tau_1, z)) dz^2 & z \in S_1 \\ \epsilon^2 P_2(\tau_2, z)^2 dz^2 & z \in S_2 \end{cases} \quad (37)$$

$$\varphi_{12} = \begin{cases} -\epsilon P_2(\tau_1, z) dz^2 & z \in S_1 \\ -\epsilon P_2(\tau_2, z) dz^2 & z \in S_2 \end{cases} \quad (38)$$

and their permutation($1 \leftrightarrow 2$). From the properties of the elliptic function with order two, it seems the number of zeros of φ does not agree with the Riemann Roch theorem. Nevertheless, on S_1 the zero of φ_{11} would appear near the pole since at zero $P_2(\tau_1, z)$ should become large in order $1/\epsilon^2$. Then it is expected that the zero would be inside of the excised disk and there are no zero of ϕ_{11} on S_1 . Probably it is because around zeros the higher order contribution becomes significant.

In the limit of $\epsilon = 0$, φ_{ij} becomes homogeneous or vanishes as well as the case of torus. The existence of the pole at $z = 0$, may seem a critical break down of the expansion. Nevertheless $z = 0$ is not included in the sewn Riemann surface since the small disk around the pole is excised in order to sew together the two tori. Furthermore also $1/\epsilon^2$ contribution of P_2 arises near the points sewing. Then the expansion seems to fail. Since the formulation of $\omega(u, v)$, however, is established in full order of ϵ we can justify the result where $|z|$ is much larger than ϵ , and P_2 is sufficiently small there. The fact that the expansion fail around the throat, will indicate the real value of φ_{ij} is unsuited for describing by ϵ expansion.

Now one may attempt to interpret the geometrical meaning of (37) and (38). If the Riemann surface is affine stretched along φ_{11} , in zero-th order torus S_1 is deformed uniformly as known in Teichmüller deformation of torus, and the other torus is not deformed. In second-order both tori are deformed in complex way. On the other hand by φ_{12} , the two tori are similarly deformed. Since the elliptic function P_2 is composed of the second order pole and its summation of images by Fuchsian group of the torus. Near region around the throat, we may account only the contribution by the pole at the center of sewing coordinate. Since the phase of φ_{ij} is hard to understand geometrical role without concrete calculation, in the present status, we will only say throat is highly deformed by the blow up of the elliptic function P_2 .

A remarkably interesting point is how the pinching parameter ϵ behaves on the Riemann surfaces being stretched. Nevertheless, ϵ which is one of the complex parameters of Teichmüller space[8] should

be expressed by the coefficients for φ_{ij} and must be calculated again from ν_i . Besides, since near the throat the contribution from the elliptic function P_2 blow up by $1/\epsilon^2$, this description will be becoming worse for the large amount of deformation around the throat. That would imply that the redefined pinching parameter become larger so that the expansion fails. For example, $\varphi_{12} \sim \epsilon$ implies the deformation by φ_{12} largely influence to ϵ .

As anticipated from these discussions, the expression may become failure around the throat of the Riemann surface. It should be emphasized that the full order expression, however, would be logically possible since the full order form of the differential forms $\omega(u, v), \nu_i, \varphi_{ij}$ is given by Yamada[10]. Then somewhere we may see the higher order effect becomes larger than the leading order. That does not mean the breakdown of formulation rather it is not effective to describe such a peculiar part of the Riemann surface. I also comment that other types of sewing of the Riemann surface analyzed by Yamada may be useful for analyzing such a peculiar part.

4 Standard constant curvature metric

4.1 Teichmüller deformation and dynamics of geometry in Einstein gravity

By the ADM canonical formulation, a dynamical description of Einstein gravity becomes possible. In (2+1)-dimensional gravity, spatial two-geometry would be variables of gravitational dynamics while the diffeomorphism invariance of spacetime diminishes its number of degrees of freedom. For example, in York's timeslicing which is a family of constant mean curvature surfaces, an assumption of homogeneous pure gravity (possibly including cosmological constant) implies that it is consistent to geodesic slice. Furthermore, one may simply choose vanishing shift vector. The conformal factor should be determined by solving Hamiltonian constraint and the momentum constraints force to extract a transverse traceless part of extrinsic curvature. Then we treat an equivalence class of two-metric by conformal transformation and diffeomorphism as the degrees of freedom of gravitation.

From the viewpoint of the Riemann surface, the conformal and diffeomorphism⁴ class of two-metric h_{ab} , is just the element of the Teichmüller space

$$\frac{\{h_{ab}\}}{\{\text{Conf.}\}\{\text{Diffeo.}\}}.$$

With the above mentioned gauge fixing, its extrinsic curvature K_{ab} and two dimensional Ricci scalar $^{(2)}R$, satisfies the constraints for homogeneous universe which is topologically genus g compact surface in pure gravity (and cosmological constant),

$$\tilde{K}_{;a}^{ab} = 0, \quad (39)$$

$$\tilde{K}_{ab}\tilde{K}^{ab} - \frac{t^2}{2} - {}^{(2)}R + 2\Lambda = 0, \quad (40)$$

where $t = -K_a^a$ and $\tilde{K}^{ab} = K^{ab} + \frac{1}{2}h^{ab}t$ is the negative trace and the trace less part of the extrinsic curvature K^{ab} , respectively.

Since the two-geometry is locally conformally flat, we are allowed to adopt the complex coordinate (z, \bar{z}) for $h_{ab}dx^a dx^b = |\Omega(z, \bar{z})|^2 dz d\bar{z}$. Then the momentum constraint can be read as $\partial_{\bar{z}}\tilde{K}_{zz} = 0$ and is considered to define a holomorphic quadratic differential of $\tilde{K}_{ab}(z)(dz)^a(dz)^b$. As a holomorphic quadratic differentials φ_{ij} ($i, j = 1, \dots, g$) spans $3g-3$ dimensional complex vector space $A_2(R)$ [12], we might expand $\tilde{K}_{ab}(dz)^a(dz)^b$ as $\sum_{i,j} P_{ij}\varphi_{ij}/2v$ with $v = \int d^2x \sqrt{h}$ for convenience.

Indeed, introducing the Veil-Peterson metric $G_{(ij)(kl)}$ [11, 12] into the Teichmüller space of torus, the Hamiltonian constraint can be rewritten for vacuum pure gravity[5],

$$\sum G_{(ij)(kl)} P_{ij} P_{kl} - v^2 t^2 = 0, \quad (41)$$

where P_{ij} is not dependent on the coordinate of the spatial hypersurface.

⁴'Diffeo' should be assigned rather by 'Diffeo₀', since the modular transformation is not included in this description.

Besides, the Riemann Roch theorem suggests φ_{ij} has $4g - 4$ zeros. In [5], the authors emphasized that the zeros imply inconsistency between these discussions and the York's slice, which means spatially constant curvature hypersurface in (2+1) dimensional pure gravity. The value of \tilde{K}_{zz} (\sim holomorphic quadratic differential) for these discussion, however, is not independent from the choice of local complex coordinate. Actually, there always exists a local complex coordinate $z \sim \int \sqrt{\varphi}$, ($\forall \varphi \in A_2(R)$)[13], to make the value of the quadratic differential φ constant though such a coordinate cannot be extended, in general, to the whole of the surface. The local coordinate is known to be related to the horizontal foliation in global. The existence of the zeros makes essential sense for this discussion as this coordinate cannot cover the whole surface prevented by the zeros of φ as illustrated in Figure 1. Since the first term of (40) is independent of the coordinate choice, the inconsistency is not recognized globally.

These aspects may be clarified by considering a homogeneous standard metric for the Riemann surface. From the uniformization theorem[11], if a two dimensional Riemannian geometry is a constant curvature, that will be realized as a fundamental region on a topologically trivial homogeneous space which is a sphere, plane and hyperboloid for positive ($g = 0$), flat ($g = 1$) and negative ($g > 1$) curvature, respectively. In other words, a Riemann surface admits any homogeneous constant curvature metric tensor. For example, the conformally invariant representative $\tilde{g}_{ab} = g_{ab}/\sqrt{g}$ of a Riemann surface and such a homogeneous metric is conformally isometric.

On the other hand, the fact that the number of solutions of the momentum constraints is finite, means they are the global degrees of freedom of (2+1)-dimensional gravity and corresponding to the deformation of the form of the homogeneous universe. Such a global deformation will be simply incorporated to the analysis as a change of boundary condition of coordinates, which is the deformation of the fundamental region, for example, in an isothermal coordinate. Otherwise, one may treat it as quasiconformal mapping of the surface, which is described by the Beltrami coefficient for its pull back of metric.

From the Teichmüller theorem[11, 12] we see any metric tensor including the information of given complex structure related to the quasiconformal mapping can be generated by the affine stretching eq.(2), keeping the boundary condition of the coordinates in non isothermal metric. One may think existence of variational holomorphic quadratic differential is inconsistent again to such a homogeneous geometry, since for the homogeneous two-geometry the first term of eq.(40) is forced to be constant. Nevertheless that does not turn out for \tilde{K}_{zz} to be constant there because it is not function but differential.

Here we will develop a homogeneous standard two-metric incorporating the information of the Teichmüller deformation. Since the time variation of the two-metric includes that of the holomorphic quadratic differential, the evolution of the complex structure of the Riemann surface will be caused by the dynamics of Einstein gravity. That will be generalized to the fact that the degrees of freedom is represented by the standard form of metric after fixing the gauge as follows.

By the deformation of the complex structure, the global aspects of the Riemann surface are changed. Since the degrees of freedom will be included into the homogeneous metric as the ellipticity or 'anisotropy' (which is called Beltrami coefficient by mathematicians), the treatment is similar to the Bianchi type universes[14]. Considering the coordinate transformation on each timeslice so as to preserve the atlas, that is the coordinate mapping on the Riemann surface, pull back of the metric can include such a mobility of the Teichmüller deformation.

By the affine stretching eq.(2) we have seen that the local geometry presenting the degrees of freedom of the complex structure. Furthermore as known in the case of torus, if the conformally invariant representative (3) admitted on the constant curvature surface, the metric

$$dl^2 = |\Omega(\zeta, \bar{\zeta})|^2 |d\zeta + k \frac{\bar{\varphi}}{|\varphi|} d\bar{\zeta}|^2 \quad (42)$$

where $\varphi \in \langle \varphi_{ij} \rangle_{\mathbb{C}} = A_2(R)$ is holomorphic quadratic differential, can become that of constant curvature by the aid of conformal factor $\Omega(\zeta, \bar{\zeta})$, and then the Hamiltonian constraint would be reduced to the independent of the spatial coordinate.

Then the spacetime geometry is examined in a synchronous reference system $-dt^2 + h_{ab}dx^a dx^b$ and

the Einstein equations on the homogeneous timeslice will be given by

$$\sum_{i,j,k,l} G'_{(ij)(kl)} P_{ij}(t) P_{kl}(t) - V(t) = 0, \quad (43)$$

$$\frac{d}{dt} P_{ij} = F_{ij}[P_{kl}(t), t], \quad (44)$$

with $\varphi = \sum_{ij} P_{ij} \varphi_{ij}$, $P_{ij} \in \mathbb{C}$, where F_{ij}, V are appropriate functionals for general relativity and $G'_{(ij)(kl)}$ are related Veil-Peterson metric. That would serve as the method to develop the dynamics of the Riemann surface in the general relativity.

4.2 constant curvature metric

To identify the dynamical degrees of freedom, it is direct way to give a standard metric containing above mentioned global degrees of freedom of the Riemann surface. We have seen how a complex structure determines the conformal class of two-geometry on Riemann surface, which will be treated in (2+1)-dimensional general relativity. Nevertheless, there is an ambiguity of Riemannian metric allowed on the Riemann surface. We know not only one (even up to local coordinate transformation), homogeneous metric exists and it is convenient to understand its characteristics, since the complex structure separately contained into homogeneous anisotropic (non isothermal) metric and the change of global boundary condition of the complex coordinates (\sim atlas of local complex coordinate). ‘From now on’, we suppose that the boundary condition of the coordinates is trivial and static, then the dynamical degrees of freedom is not contained in it but only included into the homogeneous anisotropic metric for non isothermal coordinate as the Beltrami coefficient.

Now we investigate the homogeneous standard metric of the double torus ($g = 2$) in the leading order of the pinching parameter ϵ . First of all, we prepare 0-th order background geometry of a double torus with vanishing pinching parameter ϵ . From the construction of the sewn Riemann surface, it can be easily found that it will be divided into two punctured tori.

For a later convenience, we realize them with a homogeneous geometry. Then the components become tori with a point at infinity with negative constant curvature since its Euler characteristic $\chi = 0 - 1 = -1$ is negative. From the uniformization theorem[11, 12], the universal cover of such a punctured torus is conformally equivalent to not \mathbb{C} but Δ (Poincaré disk model) or \mathbb{H} upper half space. On Δ with homogeneous metric $g_P = \frac{4dzd\bar{z}}{(1-z\bar{z})^2}$, the fundamental region \mathbf{D} of the punctured torus is a region surrounded by four circles passing through $z = \pm \frac{1}{\sqrt{2}} \pm i \frac{1}{\sqrt{2}}$. Since they are perpendicular to the sphere at infinity ($|z| = 1$), they are geodesics crossing at the infinity, (see fig. 3). Opposed edges are identified to form a torus and then it carries a complex structure of a punctured torus.

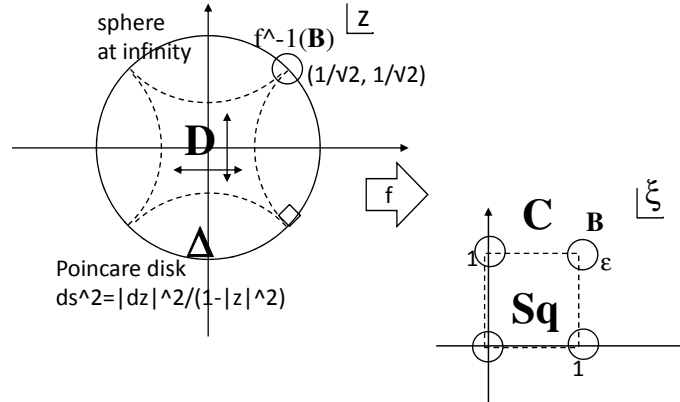


Figure 3: A square \mathbf{S}_q on \mathbb{C} is to be truncated by sewing of pinching parameter ϵ and become \mathbf{S}'_q . A holomorphic function f on \mathbf{D}' determines a conformal mapping $\xi = f(z)$, $\mathbf{D}' \mapsto \mathbf{S}'_q = f(\mathbf{D}')$.

Though we should obtain a unique bilinear two form there, it is convenient to consider ‘conformal mapping’ $\xi = f(z)$ by which the region $\mathbf{D} \subset \Delta$ is mapped to a region $\mathbf{S}_q = \{0 \leq \Re \xi \leq 1, 0 \leq \Im \xi \leq$

$1\} \subset \mathbf{C}$. Here we should be careful that a disk \mathbf{B} in \mathbf{C} , with radii $|\epsilon|$ and $f^{-1}(\mathbf{B})$ in Δ , will be excised in sewing process. Rigorously, a truncated \mathbf{D}' is conformally mapped to $\mathbf{S}'_{\mathbf{q}} = f(\mathbf{D}')$, though $|z| = 1$ implies f is not holomorphic, there.

In the coordinate ξ , pull back f^*g_P of metric g_P is given, and the quadratic differential is determined along the analysis in previous section. Then the affine stretched metric is given by

$$ds^2 = \frac{4}{(1 - z\bar{z})^2} \left| \frac{\partial z}{\partial \xi} \right|^2 \left| d\xi + k \frac{\bar{\varphi}}{|\varphi|} d\bar{\xi} \right|^2 \quad (45)$$

$$\varphi = \sum_{i,j} Q_{ij}(t) \varphi_{ij}(\xi) \quad (46)$$

$$Q_{ij}(t) \in \mathbb{C}, \varphi_{ij} \in A_2(R) \quad (47)$$

Of course, inhomogeneous φ implies the metric is inhomogeneous. In order to compensate the inhomogeneity, we consider another conformal transformation of the metric by $\tilde{g}_{ab} = |\Omega(\xi, \bar{\xi})|^2 g_{ab}$ where $\Omega(\xi, \bar{\xi})$ should satisfy the homogeneity equation (c.f., in 0-th order φ is homogeneous). As $z(\xi)$ is the bi-holomorphic transformation, $\Delta_{\xi, \bar{\xi}} \ln \frac{\partial z}{\partial \xi} = 0$, and $|\varphi| |d\xi + k \frac{\bar{\varphi}}{|\varphi|} d\bar{\xi}|^2$ is pull back of $dz d\bar{z}$, it turn out that $|\Omega|^2 = |\varphi|$ makes (45) constant curvature,

$$\tilde{ds}^2 = \frac{4}{(1 - z\bar{z})^2} \left| \frac{\partial z}{\partial \xi} \right|^2 |\varphi| \left| d\xi + k \frac{\bar{\varphi}}{|\varphi|} d\bar{\xi} \right|^2. \quad (48)$$

That would be a standard metric as another representative of an equivalent class of metric by conformal transformation and diffeomorphism of coordinate transformation.

Of course, one can easily confirm that the metric satisfy momentum constraint by checking transverse traceless component of extrinsic curvature will be proportional to the holomorphic quadratic differential φ , since that is covariant under spatial coordinate transformation corresponding to the affine stretching.

Here it should be noted that there is a ‘gauge freedom’ in conformal mapping, since holomorphic function f s.t. $f(\mathbf{D}') = \mathbf{S}'_{\mathbf{q}}$ allows still arbitrariness. While different holomorphic function f' gives another pull back f'^*g_P of metric g_P which is conformally equivalent to f^*g_P , the determined quadratic differentials are same. Consequently the resultant standard metric is equivalent to that of f up to coordinate transformation since the uniformization theorem indicates a Riemann surface admits a unique diffeomorphism equivalent class of a homogeneous geometry. The gauge would be fixed for any convenience of calculation or we will examine any gauge invariant variables like a transverse traceless component of the extrinsic curvature, which will be independent of ‘ f ’ except for its condition supposed here.

Then we can set $3g - 3$ variables $\sim Q_{ij}$ and its time derivatives as initial values. Consequently, we see there are $6g - 6$ degrees of freedom contained in eq.(48).

5 summary and discussions

Dynamical degrees of freedom of a higher genus Riemann surface universe in (2+1)-dimensional gravity has been investigated. A homogeneous standard metric that is a representative of equivalent class by conformal transformation and diffeomorphism of Riemannian metrics on two dimensional surface, have been determined in the leading order of the pinching parameter. Then we see that $6g - 6$ degrees of freedom is contained in the standard metric. Especially for $g = 2$ Riemann surface, that is so called double torus, we have made a concrete calculation which provides that up to diffeomorphism. Such a homogeneous metric will make it possible to formulate the dynamics of Teichmüller deformation in the context of ADM-formalism of pure (2 + 1) dimensional general relativity. Though only for double torus we demonstrated to have such a homogeneous standard metric, extension to general Riemann surfaces is straightforward since they are decomposed into tori to be sewn in pinching parameters.

Furthermore though we have carried out the analysis only in the leading order of the pinching parameter, a similar analysis will be possible since in Yamada’s analysis[10] the differential structure of such a sewn Riemann surface has been given in the expanding series of the pinching parameter.

As shown in the present article, the result is in approximation under the small pinching parameter ϵ . The existence of the error would indicate that the momentum constraint is not exactly satisfied. That implies the present result is not exactly covariant under coordinate transformation. Then further investigation about the higher-order analysis will reveals correct bi-holomorphic aspects of complex structure of sewn Riemann surface by the pinching parameter.

In our forthcoming work[15], the whole of the Einstein equations will be solved and reformulated for canonical formulation. Moreover the Teichmüller deformation along the holomorphic quadratic differential should be visualized, for example, in the relation with the horizontal (or vertical) foliation. It requires organized analysis and consideration by concrete complex analysis concerned with the elliptic functions. The result would be reformulated independent of gauge freedom ‘f’ for convenience, for example transverse traceless part of extrinsic curvature, or any variable in Teichmüller space. Especially, the relation between the variables introduced in this analysis and the periodic matrix which is usually used in formulation of conformal field theory in the higher genus Riemann surface.

In a range of the present investigations, we have never concerned the modular transformation of the Riemann surface, while that is mathematically well established [11] as well as that of torus is well known. For example, on considering quantum theory, that will be very important[5]. We will be also approach that in detail in forthcoming study, and that will be helpful for investigation about quantum summation of topological degrees of freedom[7][8].

From analytical reasons, the elliptic function plays essential role in the present investigation. On the other hand, some elegant algebraic theory for the elliptic function, e.g, imaginary multiplication, have known and established. Through studies of the gravitational theory of the Riemann surface, one may expect that such a mathematical knowledge provides entirely new progresses of especially quantum theory of gravitation.

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